Integration: Reversing traditional pedagogy

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Introduction

The Ancient Greeks knew something about integration. They did not have our modern concept of function, were uneasy with limits, and they certainly did not know about antiderivatives or fundamental theorems. Nevertheless, Antiphon and Bryson calculated the area of a circle by filling it up with a sequence of triangles. Archimedes did the same for the branch of a parabola. Archimedes found the volume of a sphere by considering the areas of circular cross sections and somehow adding them up. Eudoxus had previously done the same for a cone. Archimedes calculated the surface area of spheres as well. By any reasonable measure, that is integration.

All of these people lived between the fifth and third centuries BCE. Many historical references are available for these facts. We just mention Chapter 10 of Smith (1958), because it contains much of what we need in one place, and O'Connor and Robertson (2008), because it is recent, accessible and comprehensive.

The area of a rectangle is the product of its length and width. A rectangle is also the region under the graph of a constant function. Integration is thus an abstraction of multiplication.

Mankind has been differentiating for less than 350 years, since the work of Newton and Leibniz in the seventeenth century. Why did it take more than two millenia to take the leap from integrals to derivatives? Perhaps because the derivative is a deeper concept than the integral.

A derivative is the limit of a quotient. It is an abstraction of division. Since division is harder to understand than multiplication, we teach it later, hopefully only after a sound understanding of multiplication has been attained.

For the same reason, it may make sense to teach integration first, and move on to differential calculus only after a sound understanding of integration has been attained. My own thoughts on this issue are constantly changing, and I hope that others will take up this discussion. The next section describes what led me to this train of thought.

Rationale

I attempted to teach integrals before derivatives in a first calculus unit intended primarily for engineering students at the University of Ballarat in 2005. This note describes how that developed with some anecdotal comments about the students' reactions. My original motivation for this approach was purely to coordinate my teaching with the other engineering units. The engineering students study statics in their first semester and dynamics only in the second semester. Thus they need integrals (to calculate moments and centroids) well before they need derivatives (to define velocity and study kinematics). Only later did it dawn on me that this reversal of order could have educational advantages for the teaching of calculus generally. The next section describes how the unit was structured.

Integrate first!

I began with the problem of calculating areas. The area of a rectangle is not a controversial topic, and any triangle is half a rectangle, so the area of any triangle is quickly defined. Any polygon can be dissected into triangles, so we can calculate their areas too. Life gets harder when we have a region with a curved boundary. The most obvious such example is a circle.

We all get taught about πr^2 at school, but as a fact from on high. To calculate this area in the classroom, we inscribed polygons inside the circle, much as Antiphon did. First consider n triangles, each with one vertex at the centre of the circle, and the others a consecutive pair from n equally spaced points on the circumference. The sum of their areas must be less than the area of the circle. Now expand the triangles a bit so that midpoints of their third sides all lie on the circumference. The sum of their areas must exceed the area of the circle. With an estimate from both sides, we let n go to infinity — and bingo!

Not quite so fast. In the middle of this calculation, we need to know what happens to $n\sin(2\pi/n)$ as n gets larger. In order words, we are forced to introduce the concept of a limit, and to calculate the classic

$$\lim_{x\to 0}\frac{\sin x}{x}$$

I emphasised the importance of this limit and that the fact that we would need it again later. It is arguable that the concept of a limit is easier to grasp in this concrete situation than in some of the artificial exercises usually performed prior to calculating derivatives.

Next comes the problem of calculating the area under the graph of some function. The integral was introduced in the usual way, as a limit of sums of the form $\Sigma f(x_i)\Delta x$. The concept was presented without a great deal of rigor. It was emphasised that grasping the definition of an integral is important not only from our theoretical point of view, but is also an important skill in engineering problems.

How do we calculate the integral of a reasonable (e.g., continuous) func-

tion f, without resorting to the cumbersome definition every time? As usual, we defined a primitive of f to be any function F so that for any a,b in the interval where f is defined, we have

$$\int_a^b f = F(b) - F(a)$$

This formula reduces the problem of finding an area to that of evaluating a (different) function. But it begs two questions: given f, does such an F exist, and if so, how do we find it?

In one sense, the existence of a primitive is obvious: just define

$$F(x) = \int_a^x f$$

This is enlightening for some students, and confusing for others. As in any calculus course, a sound grasp of function notation is very useful at this stage.

Note that we still have not used the "D-word" in the classroom.

To be practical, the next step was to calculate the primitive for power functions. To prove that a primitive of $f(x) = x^n$ is

$$F(x) = \frac{x^{n+1}}{n+1}$$

we need some classical summation formulae, which tell us that the sum $1^n + 2^n + 3^n + \ldots + k^n$ is equal to

$$\frac{k^{n+1}}{n+1}$$

plus some lower powers of k. (We presented without proof the exact formulas for this sum for all values up to $n = 10$, as in Weisstein (2008). Most students had some memory of the formula for $n = 1$, and some had seen it for $n = 2$.) Plug this into the definition of the integral, and the formula

$$\int_a^b x^n dx = \frac{1}{n+1}\left(b^n - a^n\right)$$

then drops out after a short calculation.

We note here that this calculation was performed by Bonaventura Cavalieri and by Gilles Roberval in the 1630s, several decades before the development of differential calculus.

Next we calculated the primitive for the sine and cosine functions. The identity

$$\cos(x)+\cos(2x)+\cos(3x)+\ldots+\cos(nx) = \frac{\sin\left(\frac{1}{2}nx\right)\cos\left(\frac{1}{2}(n+1)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

can be proved easily using complex exponentials. The left side is the real part of

$$e^{ix} + e^{2ix} + e^{3ix} + \ldots + e^{inx}$$

and the result follows by applying the formula for a finite geometric series. Not assuming any knowledge of complex numbers, we just presented it as a known fact. Plugging into the definition of integral, we quickly saw that –sin is a primitive of cos. A similar calculation showed that cos is a primitive of sin.

This calculation was also performed by Roberval, long before the work of Newton and Leibniz.

One may argue that all these summation identities are unlikely to be used elsewhere in the syllabus and that their introduction is a refined form of torture. I would point out that the “D-word” has still not been used, and that the extra time spent on developing integration slowly leads to a deeper understanding.

Eventually derivatives have to be introduced, and the calculus was presented in a conventional manner after this point.

Reactions

I made no attempt to scientifically survey the opinions of the students. A straw poll at the end of the course suggested that mature age students, particularly those with no previous exposure to calculus, found this approach easier to follow. On the other hand, students straight out of Year 12 did not like it because it was not what they were used to.

This year I taught the unit again and conducted another unscientific poll at the beginning of the semester. To the question, “Which concept is easier to understand: the area of a circle or the velocity of a projectile?”, a clear majority responded, “The area of a circle.” This suggests integration is easier to grasp, and should be taught first.

However, in answer to the next question, “Which is harder to do: integral or differential calculus?”, a clear majority responded, “Integral.”

I tentatively draw two conclusions from this minuscule bit of evidence. Firstly, students’ failure to see a connection between the two questions suggests they are more concerned with getting the right answer than understanding what they are doing. (For some discussion of the relative importance of the algorithmic approach to learning and the conceptual approach, see Pettersson and Scheja (2008).) Secondly, the formal manipulations required for high school differential calculus questions are simpler than those for integral calculus.

References

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